Survey of Methods for Exhaust-Nozzle Flow Analysis

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Introduction

FOR accurate prediction of the range, payload, and operational economy of conventional, V/STOL, and supersonic jet aircraft, accurate knowledge of the velocity and discharge coefficients of the exhaust nozzle and the variation of these coefficients with nozzle pressure ratio is required. The need for high accuracy has become increasingly important because of the higher gross-to-net thrust ratios of today's high-bypass ratio turbofan engines and tomorrow's advanced supersonic jet engines. These high gross-to-net thrust ratios have the effect of greatly magnifying errors made in the prediction of the nozzle velocity coefficient. For example, studies of a Mach 2.2 supersonic transport aircraft conducted at the Douglas Aircraft Company[1] have shown that a 1% variation in the nozzle thrust coefficient results in a 3.1% change in the direct operating cost of the airplane and a 2.37% change in the specific fuel consumption.

Thus, the analytical calculations that are used to provide initial design information, to guide the modification of existing nozzles, and to identify geometries for detailed experimental testing must be as accurate as possible. In addition, the wide variety of nozzle configurations that are now being considered dictates that the analytical methods must be flexible in the geometries that they can handle, and economic constraints dictate that the methods must be fast from the standpoint of computer time.

In order to have a chance of developing an analytical method with good computational economy, it is necessary to exclude from consideration those phenomena which exercise a secondary effect on nozzle performance. Therefore, unless specifically mentioned, the methods considered in this paper will be restricted to those capable only of solving the problems of two-dimensional (planar or axisymmetric) isentropic (inviscid and shock-free) flow of a perfect gas.

Because of the high pressure ratios at which modern jet engines operate, it is necessary that analytical methods be capable of handling mixed flows, that is, flows in which both subsonic and supersonic flow regions are present. Methods that are capable of treating such problems are commonly called transonic flow methods because there are regions where the flow is sonic and near sonic. Because nozzle analysis methods must have this capability, the method of characteristics will not be discussed (since it is limited to supersonic flows), nor will methods that are appropriate only for subsonic flows (such as the methods of classical hydrodynamics).

Despite the mixed (subsonic-supersonic) nature of the flowfield, much progress has been made in recent years in the development of methods of solution appropriate for propulsion nozzle analysis. As an indication of this, the recent bibliography of Newman and Allison[2] can be cited which, although restricted to external transonic flow problems, contains over 650 entries. Indeed one finds, as eloquently expressed by Murphy[3] that "methods of obtaining numerical solutions of the equations of fluid flow have proliferated to the point where the number of different methods nearly equals the number of active workers in the field." Clearly against such a background, it is impossible to pretend that the survey presented here is complete. It is not the intention nor even the inclination of the authors to present a complete listing of all transonic nozzle analysis methods. The purpose of this survey, rather, is to classify and present critical com-

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ments regarding the accuracy, speed, and computational limitations of those methods known to the authors in order to provide an orderly framework upon which existing and future methods can be interpreted and to identify those problems for which a particular method of solution is well suited. (Just as it is impossible to list all methods of solution, it is impossible to present those methods cited in anything approaching complete detail. For this the reader is referred to the original paper.)

In this survey, the methods cited will be divided into two groups: indirect methods and direct methods. Indirect methods (or inverse methods) involve the specification of a velocity (or pressure) distribution along some reference line in the flowfield (usually the centerline). Then employing an appropriate analytical or numerical procedure, a contour which is taken to represent the opposite wall is calculated. In this way the geometry that is required to produce this desired velocity (or pressure) distribution is obtained. This procedure, in which the boundary geometry is a result of the computations, is sometimes called the design method.

In direct methods, the boundary geometry (nozzle wall) is specified and remains unchanged during the course of the calculations. The resulting solution then represents a complete description of the flowfield for the geometry under consideration. This method is sometimes called the performance method.

Indirect Methods

Perhaps the best known of the indirect methods is that of Hopkins and Hill.[4] The method is based on Friedrichs' work, in which the flowfield is developed for a prescribed velocity distribution along the nozzle axis. An interchange of the dependent and independent variables is used and series expansions for the new dependent variables are expressed as polynomials in terms of the stream function. The equations of continuity and the condition of irrotationality are then employed to obtain the coefficients of those polynomials and the resulting nozzle geometry.* A feature of this method is the introduction of a "reference boundary" and certain empirical correlations to guide the choice of the coefficients in the polynomials in order to assure that the nozzle resulting from the calculations is a good approximation of the desired nozzle geometry. It is claimed that this method is accurate for nozzles of radius of curvature ratio as small as 0.25. This method was extended by Hopkins and Hill[6] and corrected by Morden and Farquhar[7] to permit the treatment of nozzles with centerbodies. In applications of this method to nozzles of small throat radius of curvature reported by Cuffel, et al.[8] it was found necessary to introduce modifications to provide a better fit to the desired nozzle geometry in the region upstream of the nozzle throat. (The significance of the proper treatment of inlet geometry for nozzles of small radius of curvature is discussed under series expansion methods).

Another indirect method is that of van Tuyl[9] in which the centerline velocity (or Mach number) is expressed as a polynomial in terms of the axial coordinate. Pade fractions were used to produce and aid the convergence of the resulting power series which were developed for the sonic line and the limiting characteristics.

Pratt and Whitney[10] has developed an indirect method in which numerical finite-difference techniques are used to obtain the geometry of the nozzle wall in contrast to the analytical techniques used in the methods just described. An iteration procedure was built into the program to successively adjust the centerline velocity distribution and repeat the calculations until the desired nozzle geometry was obtained to within a specified degree of accuracy. The velocity at the centerline was expressed as a polynomianl in the axial coordinate. A numerical finite difference method was also used by Pirumov,[11] Norton,[12] and Ferri and Dash.[13] Pirumov investigated quadratic, arc-tangential, and exponential representations of the velocity along the axis whereas Norton investigated a velocity distribution expressed in terms of the hyperbolic tangent. In contrast to methods presented thus far, the method of Norton, as well as that of Ferri and Dash, permits rotational effects to be considered. This is accomplished by using the equations of conservation of momentum and energy in place of the condition of irrotationality. The method of Ferri and Dash is unusual in that the majority of the calculations were carried out by hand and the axial velocity distribution was prescribed tabularly and not with the use of a polynomial. This velocity distribution was successively adjusted until the desired nozzle geometry was obtained. Thrust losses from 1% to 2% were calculated as a result of the inlet flow nonuniformities considered.

The principal objection to the use of indirect methods is the potentially poorly posed nature of the problem which results from the specification of the velocity or pressure distribution along the centerline of the nozzle in the subsonic region of the flowfield. This specification is equivalent to applying Cauchy conditions along the centerline which is known to produce a destabilizing effect on the computations in the subsonic region of the flowfield as discussed in Frank[14] and Alikhashkin et al.[15] In the case of an analytical specification of the velocity distribution, such as in Hopkins and Hill,[4] for example, the Cauchy-Kowalewsky theorem indicates that a solution exists in the neighborhood of the initial data. However, as stated by Pirumov[11] there exists the possibility that "with an unfortunately chosen difference scheme the fallaciousness of Cauchy's problem may be manifested in the extremely rapid increase of round-off errors, which inevitably arise in a numerical solution."

In order to control the exponential growth of these short wave length round-off errors, particularly in repetitive indirect calculations such as in the Pratt & Whitney method,[10] it may be necessary to continually smooth the velocity distribution using, for example, a least-squares method.† In such cases it is an open question whether it is the smoothing or the physics of the problem which is driving the solution.‡ Where the velocity distribution is specified tabularly and therefore nonanalytically, the solution of the governing equations may not exist, and if it does, it may not depend in a continuous fashion on the prescribed velocity distribution.

When nonanalytical data is used to specify the velocity distribution along the axis, another problem arises in the supersonic region of the flowfield. In such cases, the zone of dependence of the difference formulation of the hyperbolic governing equations must include the zone of dependence of the original differential equations. This places a limitation on the mesh spacing and on the type of differencing method which can be employed. Further discussion of the significance of this remark will be found in the section on relaxation methods.

The destabilizing influences of the nonanalytical character of the initial data in the subsonic and supersonic regions of the flowfield can combine to preclude the attainment of a satisfactory solution. If the initial data is analytical in character and if analytical methods (in contrast to finite-difference methods) are used in the calculation procedure, useful results can, nevertheless, be obtained. The method of Hopkins and Hill[4] is an example of such a method which can be relied upon to calculate the geometry of convergent-divergent nozzles

*Serra[5] terms such methods "streamline procedures" and lists several additional references which employ this method.

†In the utilization of this method, Hamilton and Glowski[16] found that user intervention by means of an interactive computer graphics system was required in order to control the development of computational instabilities.

‡Stability can also be maintained by carrying out the calculations by hand, as was done by Ferri and Dash.[13] In this case human judgment guides the course of the solution and controls the growth of instabilities as in the interactive graphics approach of Hamilton and Glowski.[16]

corresponding to a wide variety of specified centerline velocity distributions. Free-jet boundary problems, which are encountered in convergent nozzle calculations, cannot, however, be handled by this method, and thus the effect of nozzle pressure ratio on nozzle performance cannot be calculated.

Forming a somewhat special category is the method of integral relations developed by Dorodnitsyn.[17] Polynomial expressions are used to express the velocity components in terms of the transformed transverse coordinate, and the original system of partial differential equations is replaced by a system of ordinary differential equations for the coefficients. The method was applied to nozzle flow by Holt[18] for both the indirect and direct problem. Additional investigators who have employed the method of integral relations are Belotserkovskii and Chushkin,[19] Alikhashkin et al.,[15] Favorskii,[20] Liddle and Archer,[21] and Liddle.[22]

Direct Methods

As mentioned in the Introduction, direct methods are those in which the boundary geometry is prescribed in advance and the character of the corresponding flowfield is accordingly determined. There are a bewildering array of such methods which can, with a few exceptions, be classified as either finite element, series expansion, time dependent, or relaxation methods. Before considering each of these methods in turn, some exceptions to this classification will be dealt with and grouped together as miscellaneous methods.

Miscellaneous Methods

The first methods considered in this section are exact solutions. Because the governing partial differential equations for which such exact solutions are sought are of mixed-type and nonlinear, this category is extremely small. Perhaps the best known is the solution of Ringleb.[23] Ringleb's solution is conventionally interpreted as describing compressible flow around a 180° turn; however, the streamlines resemble the walls of a family of convergent-divergent nozzles. If a pair of such streamlines is selected, the flow between them can be taken to represent the flow through a curved convergent-divergent nozzle. Recognizing this fact, Douglass[24] calculated the discharge coefficient, velocity coefficient, and optimum nozzle performance for a family of such nozzles. Other exact solutions for geometries of interest for propulsion applications are unlikely to be forthcoming in the foreseeable future due to the difficult nature of the governing equations.

Despite the dearth of exact solutions for transonic flow in propulsion nozzles, many exact solutions do exist for incompressible flow in nozzles having possible propulsion applications. It is natural, then, to seek approximate methods for adjusting these solutions to account for compressibility effects. An interesting thermodynamic approximation gives rise to a method of this type that has been used to solve mixed-flow nozzle problems. It is based on Chaplygin's discovery[25] that if the isentropic relationship between pressure and specific volume is approximated by a straight line, the governing equations of planar compressible flow in the hodograph variables become those of incompressible flow in transformed hodograph variables. This approximation (known also as the tangent-gas or Karman-Tsien approximation) has been used by Von Busemann,[26] Prince,[27] and Brown.[28] Prince and Brown were concerned with the solution of convergent nozzle problems, in which the value of the nozzle pressure ratio influences the nozzle performance. Such convergent nozzle problems are inherently more difficult than convergent-divergent nozzle problems because of the need to calculate the shape of the supersonic jet which is produced at the nozzle exit. Prince and Brown solved this problem by matching the solution of the subsonic flowfield that was obtained by the tangent-gas approximation to the solution of the supersonic flowfield that was obtained by the method of characteristics. The tangent-gas method, however, suffers from its limitation to planar flows and from the arbitrary way in which the point of tangency is chosen.

If one accepts being restricted to planar flows, an interesting matching solution for transonic nozzle problems can be developed which does not require the approximation described above. In this method, typified by the work of Norwood[29] and Benson and Pool,[30] the governing equations for the subsonic region are expressed in terms of the hodograph variables and solved by standard relaxation methods. The use of hodograph variables permitted easy matching with the supersonic solution which was obtained with the method of characteristics. Such matching solutions, however, are not mathematically unique because of the arbitrary selection of the matching condition and are inherently inefficient as a result of the repeated calculations which are necessary in order to match the subsonic and supersonic solutions.

Streamline curvature (or stream filament) methods are another category of miscellaneous methods. These are methods in which a knowledge of the curvature (or radius of curvature) of the streamlines and the condition of irrotationality are used in conjunction with the integral form of the continuity equation to predict the velocity distribution along the solid boundaries of the flowfield. In its original form developed by Perl,[31] the method was applied to isolated airfoil problems, and an empirical relationship for the radius of curvature based on a solution for incompressible flow past the same airfoil was used. The use of streamline curvature information from a corresponding incompressible solution is characteristic of these methods. This was also the approach of Katsanis[32] who used the streamline curvature method to investigate cascade flows. Using the incompressible streamlines, Katsanis obtained a relationship between the weight flow through the cascade and the velocity distribution on the upper and lower blade surfaces. A similar method has been developed by Bindon and Carmichael,[33] Frost,[34] and Wilkinson.[35] The disadvantage of the streamline curvature method is that for transonic and supersonic flows the correspondence between the incompressible and compressible streamlines is poor, and thus the accuracy of the method is compromised.

The final category of methods considered in this section are error-minimization methods. They are similar in concept to the Galerkin-type finite-element methods in that a minimization of residuals is sought. However, the solution is obtained by a generalized Newton-Raphson method rather than by finding the stationary value of an integral. In addition, in error-minimization methods no local functional representation of the dependent variable is employed. This method was first applied to transonic flow problems by Prozan and Kooker.[36] Although no information on mesh size or computation time was given, the results reported were in excellent agreement with the experimental data of Cuffel et al.[8] In Prozan[37] special attention was given to nozzles of large entrance angle and small throat radius of curvature. Careful examination of this method by the authors and Glasgow et al.[38,39] revealed several undesirable features which included failure of the solution to observe conservation of mass, sensitivity of the solution to the selection of the mesh spacing, and long computational times (about the same as time dependent methods). Therefore, the authors concur with the evaluation of this method by Glasgow et al.[39] and recommend caution in its use for transonic flow analysis in its present form.

Finite-Element Methods

Finite-element methods have seen application in structural and solid mechanics problems for several years and only recently have been applied to fluid flow problems. In this method, the entire flowfield is broken up into subregions or elements (usually triangles) and in each element a functional form (usually polynomial) of the dependent variables is assumed. Then, using this functional representation of the dependent variables, the minimization of an integral is sought which represents either the variational form of the governing equations (Ritz method) or a weighted sum of the residuals (Galerkin method). This minimization is effected by taking the derivative of the integral with respect to each of the unknown dependent variables (which form the nodes of the finite element network) and equating the results to zero. This results in a set of symmetric algebraic equations which can then be solved by standard matrix inversion techniques. For a detailed understanding of the finite-element method, the interested reader should consult the textbooks by Norrie and de Vries[40] and Zienkiewicz[41] as well as a particularly clear exposition of the method which can be found in Myers.[42]

An excellent survey article by Norrie and de Vries[43] lists more than one hundred references dealing with fluid dynamic applications of finite-element methods. These references, however, deal primarily with incompressible flows. Leonard[44] was one of the first to consider compressibility effects. In his pioneering paper he solved a linerized form of the governing equations using the Galerkin method. More recently, Carey[45] used a variational method to solve the small-perturbation equations for the flow about a circular cylinder. Free stream Mach numbers high enough to produce critical flow on the cylinder were considered. Solution of a mixed-flow problem has not yet been attempted with finite element methods although Shen[46] gives some advice for constructing such a solution. He suggests that the solution be obtained by a patchwork approach in which conventional finite element methods are used in the subsonic region of the flow and "space-time" finite elements in the supersonic region of the flow. In the transonic region he recommends a "semi-discrete" procedure similar to the method of integral relations. It has recently come to the authors' attention that Chan and Brashears[47] have also suggested a finite-element method for solving mixed-flow problems. They proposed to use a "staggered two-strip" approach to solve the transonic small-perturbation equations but did not carry out any calculations.

At this writing, it seems to the authors that the advantages of the finite-element method over the finite-difference method have not been clearly established. Although it is generally conceded that finite-element methods permit complex boundary geometries to be more easily handled and permit nodal points to be easily clustered in regions where the solution changes rapidly, the literature contains conflicting claims regarding the computational efficiency and accuracy of finite-element methods. This controversy, however, is likely to be short-lived since the intensive development which finite-element methods are now undergoing will soon provide a more definitive comparison of the two methods.

Series Expansion Methods

Similar to several types of indirect methods, series expansion methods employ a polynomial representation of the dependent variables. These polynomials, however, represent the dependent variables throughout the flowfield and not only on the centerline. Furthermore, the coefficients of these polynomials are obtained analytically from the given nozzle contour rather than iteratively by successive adjustments as is done in indirect methods. An excellent survey of the early development of such methods can be found in Hall and Sutton.[48] The early methods including the more recent work of Martensen[49] and Martensen and von Sengbush[50] are primarily of historical interest because of the difficulty of evaluating coefficients of the double power series used in these solutions.

The methods of Oswatitsch and Rothstein[51] and Sauer[52]§ were developed more than two decades ago but definitely have more than historical interest. Although these methods are frequently grouped together, they are in reality quite different in approach. Oswatitsch and Rothstein employ the complete form of the governing equations whereas Sauer approximates these equations by using the transonic small perturbation method. The result of this approximation is that Sauer's method is restricted to nozzles with circular-arc throat profiles whereas the method of Oswatitsch and Rothstein in principle permits nozzles of any shape to be considered. (There is, in fact, a limitation to the throat radius of curvature for this as well as for each of the series expansion methods which will be considered at the end of this section).

Restriction to nozzles with circular throats permitted Sauer to obtain an explicit expression for the character of the flow in the throat, whereas the method of Oswatitsch and Rothstein requires that an ordinary differential equation be solved before such information can be obtained. A recent investigation by Smithey and Naber[53] used a method similar to Sauer's to obtain a solution for the flow in the vicinity of the throat of an annular nozzle with a cylindrical centerbody.

The use of the small perturbation form of the governing equations was continued by Hall,[54] Moore and Hall,[55] and Moore.[56] Rather than a series expansion in terms of the axial and radial coordinates, Hall and Moore employed a series expansion in terms of the throat radius of curvature of the nozzle. In Hall,[54] results for nozzles with throats of circular, parabolic, and hyperbolic-arc profiles are presented with terms up to third order in radius of curvature retained. The results are claimed to be an improvement over the solutions of Oswatitsch and Rothstein[51] and Sauer.[52] Moore[56] extended Hall's solution to permit planar nozzles with asymmetric profile to be considered. Two shapes that were considered as special cases were nozzles of parabolic profile having asymmetry with respect to the horizontal axis and nozzles of cubic profile having asymmetry with respect to the vertical axis. In contrast to Hall,[54] Moore employed a series expansion in terms of powers of the square root of the nozzle radius of curvature. (Moore also presents a description of an indirect method in which expressions are developed for coefficients which determine the shape of the nozzle wall corresponding to a given axial velocity distribution). In Moore and Hall,[55] Moore's solution was extended in order to enable annular nozzles with arbitrary throat profiles to be considered.

Kliegel and Quan[57] employed a method similar to that of Hall[54] except that a stretching of the axial coordinate was introduced to increase the accuracy of the solution in the subsonic and supersonic regins of the flowfield. It is claimed that the results more accurately include the effects of wall geometry than does Hall's solution.

Kliegel and Levine[58] utilized a toroidal coordinate system in order to improve the accuracy of Hall's solution for nozzles of small radius of curvature. Consequently the dependent variables are expressed in terms of powers of $R+1$, where R is the radius of curvature ratio of the nozzle. In an appendix of their paper, Kliegel and Levine correct an error appearing in the expression for the discharge coefficient in Hall.[54]

The principal disadvantage of series expansion methods arises from the small perturbation approximation which is applied to the governing equations. With the exception of the method of Oswatitsch and Rothstein[51] in which such an approximation is not employed, the small perturbation approximation has the effect of greatly reducing the accuracy of the solution in the region away from the nozzle throat. This is clearly shown by the experiments of Back et al.[59] If only the solution in the throat is desired, this effect is not of concern. However, for nozzles of throat radius of curvature ratio less than 1.5, Hopkins and Hill[4] have shown that improper treatment of the flowfield upstream of the throat can affect the accuracy of the solution in the throat. For these cases series expansion methods must be used with caution. Both as a result of this effect and because of the convergence properties of the particular series expansion used, a lower limit of the wall

§It should be noted that Hall and Sutton[48] classify these two methods as being indirect. These methods are not indirect in the sense intended here.

radius of curvature ratio can be identified for each method just described. Below this value the accuracy of these methods is expected to rapidly deteriorate. The method of Hall[54]—which was claimed to be superior to that of Sauer[52] and Oswatitsch and Rothstein[51]—is actually less accurate than these methods for radius of curvature ratios less than 5 and these methods are in turn less accurate than the methods of Kliegel and Quan[57] and Hall and Moore[55,56] for radius curvature ratios less than 2. For radius of curvature ratios less than unity only the method of Kliegel and Levine[58] has sufficient accuracy, although according to Fig. 2 of Back and Cuffel,[64] at large radius of curvature ratios this method predicts discharge coefficients somewhat higher than experiment. Summarizing these results then, it would appear that for nozzles with radius of curvature ratios greater than 2 nothing is gained by using more complicated methods than the straightforward method of Sauer.[52] For nozzles with radius of curvature ratio less than 2 the method of Kliegel and Levine[58] is most reliable.

Although series expansion methods are appropriate (with the stated limitations) for convergent-divergent nozzle problems, the effect of nozzle pressure ratio on nozzle performance cannot be calculated with these methods. This is because series expansion methods depend upon a prior specification of the entire boundary geometry. But the position of the free-jet boundary, which is necessary to calculate pressure ratio effects, cannot be specificed a priori.

Time-Dependent Methods

One of the facts which must be accounted for when numerical methods (in contrast to analytical methods such as series expansion methods) are used to solve the governing equatins of steady transonic flow is that the governing equatins are elliptic in the subsonic regions and hyperbolic in the supersonic regions of the flowfield. If the unsteady terms are added to these equations, however, the governing equations become hyperbolic throughout the flowfield. The desired steady-state solution can then be viewed as being obtained in an asymptotic sense at large time from arbitrarily assumed initial conditions. The use of time-dependent methods transforms a steady, mixed-flow boundary-value problem to an unsteady, hyperbolic, initial-value problem. The time-dependent method has been the most popular technique for solving transonic flow problems for the past ten years. For a very readable introduction to time dependent methods the interested reader is referred to Jenkins[60] and for a detailed consideration of the merits of various time-dependent methods to Richtmeyer[61] and Moretti.[62]

Saunders[63] was the first to apply time-dependent methods to nozzle flow problems. To numerically integrate the governing equations a two-step Lax-Wendroff scheme was used. The results compared well with the experimental wall static pressure data measured by Back, et al.[59] for a convergent-divergent nozzle. The computation time for Saunders' method on the CDC3200 computer was approximately 45 min. (253 mesh points).

Brunell[65] employed an upwind differencing technique rather than the Lax-Wendroff scheme to the problem of planar, two-dimensional flow through a duct with arbitrary shaped wall contour and centerbody. Irresolvable problems were encountered in predicting Mach line contours due to disturbances which were produced in the supersonic region of the flowfield. In addition, the calculated wall pressure distributions were in poor agreement with values measured in the throat of an annular converging-diverging nozzle.

Laval[66] investigated converging-diverging nozzle flows using a method similar to that of Saunders[63] except that artificial viscosity was explicitly introduced in order to stabilize the computations. In general, good agreement was obtained betwen calculated and experimental wall pressure distributions, Mach line contours, and discharge coefficients for a considerable variety of geometries. Computation time on an IBM 360/50 computer was approximately 2 hrs (1564 mesh points).¶

Simultaneously with Laval, Serra[5] was carrying out a similar investigation using the one-step Lax-Wendroff method. Excellent agreement was shown with experimental results for both convergent-divergent nozzles and annular ducts. Computation time on a Univac 1108 computer was 80 min (3000 mesh points).

One notable exception to the long computation times encountered by other investigators is the work of Migdal et al.[67] in which the one-step Lax-Wendroff scheme was used. Good agreement was obtained with the wall pressure distributions measured by Back et al.[59] for a 45-15° convergent-divergent nozzle. Calculation time was reported to be 5 min on an IBM 360/75 but no information on the mesh size was given.

Wehofer and Moger[68] used Saunders' method to investigate the effect of inlet flow nonuniformities on nozzle performance. The most interesting aspect of this work, however, is the capability of the method to handle convergent nozzle problems and the associated free-jet calculations. This enables the influence of nozzle pressure ratio on performance to be predicted. Although this influence had been considered for two-dimensional flows, this is the first time that such a method had been developed for axisymmetric flow. Good agreement with Thornock's convergent nozzle data[69] was obtained at large values of the pressure ratio, but the agreement of the predicted sonic line with experiment at low pressure ratios was disappointingly poor. In addition, the computational time for the convergent nozzle calculations is large, requiring from 2-5 hr on an IBM 360/50 (187-1400 mesh points). Brown and Ozcan[70] used the method described in Migdal et al.[67] and the treatment of the free-jet boundary adapted from the blunt-body solution of Moretti and Abbett[71] in an attempt to improve the accuracy and reduce the computational time of Wehofer and Moger's convergent nozzle calculations. The results show good agreement with the discharge coefficients of Thornock[69] and excellent agreement with the sonic line position for all values of the pressure ratio considered. The computational time amounted to less than 17 min on an IBM 360/65 (120 mesh points).

Recently Cline[72] reported calculations which were considerably faster than previous time-dependent solutions. Cline used the MacCormack[73] differencing scheme for the interior points and a reference-plane method of characteristics technique for the boundary points. Calculations for a convergent-divergent nozzle, convergent nozzle, and a plug nozzle were carried out and the agreement of the results compared with experimental data was excellent. The computational time for the convergent nozzle calculations on a CDC 6600 was 35 sec (168 mesh points).

Time-dependent calculations can be relied upon to give accurate results for a wide variety of propulsion nozzles and have a great advantage over series expansion methods in that nozzles of extremely small radius of curvature can be accurately calculated. Laval[66] considered nozzles with radius of curvature ratios as small as 0.1. The accuracy of the results is excellent not only in the throat, but as shown by Serra,[5] in the subsonic regions of the flowfield as well. In addition, the effects of shock waves and viscosity can be included in the calculations, and, unlike series expansion methods, time-dependent methods are capable of handling convergent nozzle calculations and inlet flow nonuniformities.

The overwhelming disadvantage of time-dependent methods is their excessively long computational times. This is shown in Table 1 where the computational times of the various methods discussed above are summarized, Nieuwland and Spee[75] have postulated that the inefficiency of time-dependent methods results "because of the need to represent the time history of a large amount of physical detail that must

¶These results are for a conventional nozzle. Calculations were also carried out on an annular nozzle which required more than 3 hr (1275 mesh points).

Table 1 Representative computation times for time-dependent methods

Investigator	Mesh	Machine	Time (min)	Time per point (sec)	Normalized[a] time per point (sec)
Saunders[63]	253	CDC 3200	45	10.7	N/A[b]
Laval[66]	1564	IBM 360/50	135	5.2	0.8
Serra[5]	3000	UNIVAC 1108	80	1.6	0.4
Wehofer and Moger[68]	1400	IBM 360/50	300	12.9	1.9
Brown and Ozcan[70]	120	IBM 360/65	17	8.5	2.6
Cline[72]	168	CDC 6600	0.6	0.20	0.20

[a]Normalized with respect to an IBM 370/165 with information supplied by Alper.[74] [b]Not available.

be considered to be irrelevant if the sole aim is an asymptotic steady-state solution." Murman[76] states that although direct comparisons of the same problem on the same machine with the same number of mesh points have not been made, "indications are that relaxation methods are faster by a factor of 5 to 10," with no decrease in the accuracy of the results. It is natural then to consider relaxation methods next.

Relaxation Methods

Relaxation methods are without a doubt among the most time-honored methods for the solution of two-dimensional fluid flow problems. Relaxation methods represent a systematic approach to solving the governing equations by means of successive approximation procedures which utilize assumed initial values for the unknowns. Relaxation methods were first applied to problems of transonic flow by Emmons.[77,78] Such early methods, however, were designed for hand calculations. It is interesting to note that when Emmons' method was programed on an electronic computer it suffered from convergence difficulties, and Murman[76] concluded that "apparently a human decision must be involved in the relaxation process."

Consequently, relaxation procedures for transonic flow problems fell into disrepute and were not revived until the recent work of Murman and Cole.[79] The difficulty with Emmons' method—recognized by Murman and Cole and conclusively demonstrated mathematically by Kentzer[80]—is that the centered differencing scheme is inherently unstable in the supersonic region of the flowfield. Recognizing this, Murman and Cole replaced the centered differencing scheme for axial derivatives in the supersonic region by an upwind (or backward) differencing procedure. The essential feature of the relaxation method of Murman and Cole is that the local character of the flowfield (subsonic or supersonic) is taken into account in choosing the difference formula. Of course additional logic must be built into the program to determine the local Mach number in order to accomplish this.

The relaxation scheme developed by Murman and Cole has been extensively used in airfoil calculations. This work is summarized in Niewland and Spee,[75] Brainerd and Shih,[81] and Yoshihara.[82] The computational times of these solutions were tabulated by the authors and found to be an order of magnitude less than those required for solutions of similar problems by the time-dependent method.

Recently, work was undertaken by the first author to solve the full potential equation for mixed flow in a convergent-divergent nozzle using Murman's type-dependent differencing scheme. The result of preliminary calculations showed good agreement with the series expansion solution of Oswatitsch and Rothstein.[51] The calculation time was 3.5 min on an IBM 370/185 (275 mesh points). This corresponds to a normalized time per point of 0.13 sec which compares favorably with the times required for external flow solutions mentioned above and is nine times faster than the average time-dependent method cited in Table 1.

Conclusions

It is the authors' hope that the classification and discussion of relative merits of the methods cited in this paper will bring to the attention of the nozzle designer the vast number of analytical methods available to him and will enable him to select the method that possesses the appropriate combination of accuracy, speed, and computational flexibility.

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